Parameter Bounds for Discrete-Time Hammerstein Models With Bounded Output Errors

V. Cerone and D. Regruto

***Abstract*—In this note, we present a two-stage procedure for deriving parameters bounds in Hammerstein models when the output measurement errors are bounded. First, using steady-state input–output data, parameters of the nonlinear part are tightly bounded. Then, for a given input transient sequence we evaluate tight bounds on the unmeasurable inner signal which, together with noisy output measurements are used for bounding the parameters of the linear dynamic block.**

***Index Terms*—Bounded uncertainty, errors-in-variable, Hammerstein model, linear programming, output errors, parameter bounding.**

I. Introduction

Most physical systems are inherently nonlinear, and, though in some cases they can be represented by linear models over a restricted operating range, only nonlinear representations are adequate for their description. A wide class of nonlinear systems, also called block-oriented systems, can be modeled by interconnected memoryless nonlinear gains and linear subsystems. Nonlinearities may enter the system in different ways: either at the input or at the output end or in the feedback path around a linear model. The configuration we are dealing with in this note, commonly referred to as a Hammerstein model, is shown in Fig. 1; it consists of a static nonlinear part $\mathcal{N}$ followed by a linear dynamic system. The identification of such a model relies solely on input–output measurements, while the inner signal x_t, i.e., the output of the nonlinear block, is not assumed to be available.

Identification of the Hammerstein structure has attracted the attention of many authors, as can be seen in [1] and [2]. Existing identification procedures can be roughly classified on the basis of the representation (parametric or nonparametric) chosen to model the nonlinear and the linear subsystems. As far as the estimation of the nonlinear block is concerned, in the parametric approach the nonlinearity is usually modeled by a polynomial with a finite and known order or, more generally, with a series expansion of a known basis of nonlinear functions (see, e.g., [3]–[5]). On the contrary, in the nonparametric approach no *a priori* information on the structure of the nonlinearity is assumed to be available and the mapping between the input signal and the intermediate signal might not be finitely parameterizable. Thus, in that case, only mild prior assumptions are made, e.g., continuity and piecewise smoothness [6] or membership to some very general class of functions [7]. As far as the estimation of the linear block is concerned most of the contributions use parameterized structures like autoregressive with exogenous input (ARX), finite-impulse response (FIR), or output error models while some works assume a nonparametric description based only on the stability of the system (see, e.g., [6] and [7]). Different methods were proposed in the literature to estimate the parameters of the nonlinear static block and the linear dynamic part either iteratively or simultaneously. Among the noniterative algorithms, we mention the over-parameterization method proposed by Chang and Luus in [4] which has been extended by Hsia in [5] to deal with the case of correlated noise and by Bai in [8] which provides a two stage globally optimal algorithm. Further noniterative solutions include the approach based on proper extension of subspace model identification [9], the blind approach proposed in [10] and the method proposed in [11] in which a closed form solution to the problem of minimum-variance approximation of nonlinear systems by means of Hammerstein models is presented in the case of white noise input. Furthermore, Stoica and Söderström [12] proposed a parametric instrumental variable method which, in the presence of either a strictly persistently exciting sequence or a white noise, provides consistent estimates. Iterative methods, introduced in [3], are based on the idea of alternate estimation of the parameters of the linear and the nonlinear subsystems. The main problem with iterative procedures is to prove convergence of the estimate under general conditions. In [13], it is shown that the algorithm proposed in [3] can diverge. Recently, Rangan *et al.* in [14] have proposed a modification of the standard iterative algorithm that allows the above mentioned convergence problem to be overcome, provided that the linear subsystem is FIR and the input signal is white noise. Other proposed iterative procedures are the algorithms based on Bussgang's theorem (see, e.g., [15] and [16]) and the one proposed in [17]. On the nonparametric side, most of the methods are based either on the estimation of a nonparametric kernels regression (see, e.g., [7] and [18]) or on the property of the Fourier series representation (see, e.g., [6] and [19]).

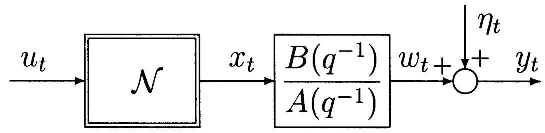

Fig. 1. Single-input–single-output (SISO) Hammerstein model.

In all of the papers previously mentioned, the authors assume that the measurement error η_t is statistically described. However, there are many cases where in practice either *a priori* statistical hypotheses are seldom satisfied or the errors are better characterized in a deterministic way. Some examples are given by systematic and class errors in measurement equipments, and rounding and truncation errors in digital devices. A worthwhile alternative to the stochastic description of measurement errors is the bounded-errors characterization, where uncertainties are assumed to belong to a given set. In the bounding context, all parameter vectors belonging to the *feasible parameter set* (FPS), i.e. parameters consistent with the measurements, the error bounds and the assumed model structure, are feasible solutions of the identification problem. The interested reader can find further details on this approach in a number of survey papers (see, e.g., [20] and [21]), in the book edited by Milanese *et al.* [22], and the special issues edited by Norton [23], [24]. To the best of our knowledge, only few contributions can be found which address the identification of Hammerstein models when the measurement error η_t is supposed to be bounded. Belforte and Gay [25] considered a Hammerstein model where the linear block is described by an ARX model. They proposed a solution through the introduction of a linearized augmented Hammerstein model (see, e.g., [4] and [8]), whose parameters are identified first using any algorithm available in the parameter bounding literature. From the parameter bounds of such a model, overbounds on both nonlinear and linear block parameters are then derived. Boutayeb and Darouach [26] proposed a recursive estimator which provides a single parameter vector belonging to the feasible parameter region defined on a suitable finite horizon time. Garulli *et al.* [27] considered the identification of low complexity approximate Hammerstein models for a class of nonlinear systems. They proposed a procedure for the computation of the

Manuscript received October 31, 2002; revised June 16, 2003. Recommended by Associate Editor E. Bai. This work was supported in part by the Italian Ministry of Universities and Research in Science and Technology (MURST), under the plan "Robustness techniques for control of uncertain systems."

The authors are with the Dipartimento di Automatica e Informatica, Politecnico di Torino, 10129 Torino, Italy (e-mail: vito.cerone@polito.it; diego.regruto@polito.it).

Digital Object Identifier 10.1109/TAC.2003.817945

Chebichev conditional center of the FPS when the noise is bounded in either ℓ_∞ or ℓ_2 norm.

In this note, we consider the identification of SISO Hammerstein models when the nonlinear block can be modeled by a linear combination of a finite and known number of nonlinear static functions, the linear dynamic part is described by an output error model and the output measurement errors are bounded. We present a two-stage identification procedure. First, parameters of the nonlinear block are tightly bounded using input–output data collected from the steady-state response of the system to a set of step inputs with different amplitudes. Then, through a dynamic experiment, for all u_t belonging to a given input transient sequence $\{u_t\}$, we compute tight bounds on the inner signal which, together with noisy output measurements are used for bounding the parameters of the linear part.

II. Problem Formulation

Consider the SISO discrete-time Hammerstein model depicted in Fig. 1, where the nonlinear block maps the input signal u_t into the unmeasurable inner variable x_t through the following nonlinear function:

$$x_t = \sum_{k=1}^{n} \gamma_k \psi_k(u_t), \qquad t = 1, \ldots, N \tag{1}$$

where $(\psi_1, \ldots, \psi_n)$ is a known basis of nonlinear functions; N is the length of the input sequence. The linear dynamic part is modeled by a discrete-time system which transforms x_t into the noise-free output w_t according to the linear difference equation

$$A(q^{-1})w_t = B(q^{-1})x_t \tag{2}$$

where $A(\cdot)$ and $B(\cdot)$ are polynomials in the backward shift operator q^{-1}, $(q^{-1}w_t = w_{t-1})$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{na} q^{-na} \tag{3}$$
$$B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_{nb} q^{-nb}. \tag{4}$$

In line with the work done by a number of authors, we assume that: 1) the linear system is asymptotically stable (see, e.g., [12], [18], and [28]–[30]); 2) $\sum_{j=0}^{nb} b_j \neq 0$, that is, the steady-state gain is not zero (see, e.g. [28]–[30]); and 3) the only *a priori* information needed is an estimate of the process settling-time (see, e.g., [31]). Let y_t be the noise-corrupted output

$$y_t = w_t + \eta_t. \tag{5}$$

Measurements uncertainty is known to range within given bounds $\Delta\eta_t$, i.e.,

$$|\eta_t| \leq \Delta\eta_t. \tag{6}$$

Unknown parameter vectors $\gamma \in R^n$ and $\theta \in R^p$ are defined, respectively, as

$$\gamma^T = [\gamma_1 \quad \gamma_2 \quad \ldots \gamma_n] \tag{7}$$
$$\theta^T = [a_1 \quad \ldots \quad a_{na} \quad b_0 \quad b_1 \quad \ldots b_{nb}] \tag{8}$$

where $n_a + n_b + 1 = p$. It is easy to show that the parameterization of the structure of Fig. 1 is not unique. As a matter of fact, any parameters set $\tilde{b}_j = \alpha^{-1} b_j$, $j = 1, 2, \ldots, nb$, and $\tilde{\gamma}_k = \alpha\gamma_k$, $k = 1, 2, \ldots, n$, for some nonzero and finite constant α, provides the same input–output behavior. Thus, any identification procedure cannot perceive the difference between parameters $\{b_j, \gamma_k\}$ and $\{\alpha^{-1}b_j, \alpha\gamma_k\}$. To get a unique parameterization, in this work, we assume, without loss of generality, that the steady-state gain of the linear part be one, that is

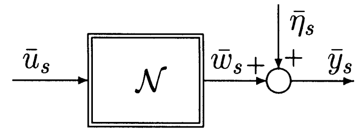

Fig. 2. Steady-state behavior of the Hammerstein model when $g = 1$.

$$g = \frac{\sum_{j=0}^{nb} b_j}{1 + \sum_{i=1}^{na} a_i} = 1. \tag{9}$$

In this note, we address the problem of deriving bounds on parameters γ and θ consistently with given measurements, error bounds and the assumed model structure. In Section III, using steady-state input-output data, parameters of the nonlinear part are tightly bounded, while in Section IV, for a given input transient sequence we evaluate tight bounds on the unmeasurable inner signal which, together with noisy output measurements are used for bounding the parameters of the linear part. A simulated example is reported in Section V.

III. Assessment of Tight Bounds on the Nonlinear Static Block Parameters

In most physical processes, we can collect a great deal of data, which often contains steady-state measurements at many different operating conditions. However, usually, only transient data are used in the identification process while steady-state measurements are not explicitly considered. Although data are assumed to be generated by a persistently exciting input, in practice a given plant might only be mildly perturbed around operating conditions, leading to a shortage of proper nonlinear information in the transient data. In this note, we exploit steady-state operating conditions to bound the parameters of the nonlinear static block. The noise corrupted output sequence is collected from the steady-state response of the system to a set of step inputs with different amplitudes. For each value of the step input amplitude, only one steady-state value of the noisy output is considered. Thus, given a set of step inputs with M different amplitudes, M steady-state values of the output are taken into account. We only assume to have a rough idea of the settling time of the system under consideration, in order to know when steady-state conditions are reached, so that steady-state data can be collected. Indeed, under conditions 1), 2), and 3) stated in Section II, combining (1), (2), (5), and (9) at steady-state, we get the following input–output description involving only the parameters of the nonlinear block:

$$\bar{y}_s = \sum_{k=1}^{n} \gamma_k \psi_k(\bar{u}_s)\bar{\eta}_s, \qquad s = 1, \ldots, M \tag{10}$$

where $\bar{u}_s$, $\bar{y}_s$ and $\bar{\eta}_s$ are steady-state values of the known input signal, output observation and measurement error respectively; $M \geq n$ is the number of the steady-state samples. A block diagram description of (10) is depicted in Fig. 2. The feasible parameter region of the static nonlinear block is defined as

$$\mathcal{D}_\gamma = \left\{ \gamma \in R^n : \bar{y}_s = \sum_{k=1}^{n} \gamma_k \psi_k(\bar{u}_s) + \bar{\eta}_s, \right.$$
$$\left. |\bar{\eta}_s| \leq \Delta\bar{\eta}_s, \qquad s = 1, \ldots, M \right\} \tag{11}$$

where $\{\Delta\bar{\eta}_s\}$ is the sequence of bounds on measurements uncertainty. From definition (11) it can be seen that $\mathcal{D}_\gamma$ is exactly described by the following constraints in the n-dimensional parameter space:

$$\bar{\varphi}_s^{\mathrm{T}}\gamma \le \bar{y}_s + \Delta\bar{\eta}_s \quad \bar{\varphi}_s^{\mathrm{T}}\gamma \ge \bar{y}_s - \Delta\bar{\eta}_s \tag{12}$$

where

$$\bar{\varphi}_s = [\psi_1(\bar{u}_s) \quad \psi_2(\bar{u}_s) \quad \psi_3(\bar{u}_s)\dots\psi_n(\bar{u}_s)]^{\mathrm{T}} \tag{13}$$

for $s = 1, 2, \dots, M$. This exact description of $\mathcal{D}_\gamma$ will be used in the next section when deriving tight bounds on the unmeasurable inner signal x_t.

Since $\mathcal{D}_\gamma$ is a convex polytope, whose shape may become quite complex for increasing n and M, an outer bound to it such as an ellipsoid or a box is often computed. In this note, we consider an orthotope-outer bounding set $\mathcal{B}_\gamma$ containing $\mathcal{D}_\gamma$

$$\mathcal{B}_\gamma = \left\{\gamma \in R^n : \gamma_j = \gamma_j^c + \delta\gamma_j, |\delta\gamma_j| \le \Delta\gamma_j, \qquad j = 1, \dots, n\right\} \tag{14}$$

where

$$\gamma_j{}^c = \frac{\gamma_j^{\min} + \gamma_j^{\max}}{2} \quad \Delta\gamma_j = \frac{\left|\gamma_j^{\max} - \gamma_j^{\min}\right|}{2} \tag{15}$$

$$\gamma_j^{\min} = \min_{\gamma\in\mathcal{D}_\gamma} \gamma_j \quad \gamma_j^{\max} = \max_{\gamma\in\mathcal{D}_\gamma} \gamma_j. \tag{16}$$

The set $\mathcal{B}_\gamma$ as defined in (14) is a tight orthotope outer-bound on the exact feasible parameter region $\mathcal{D}_\gamma$ and its evaluation requires the solution of $2n$ linear programming (LP) problems with n variables and $2M$ constraints. The significance of central estimates γ_j^c and parameter uncertainty bounds $\Delta\gamma_j$, $j = 1, 2, \dots, n$, which in turn define $\mathcal{B}_\gamma$ through (14), will be shown in the numerical simulation introduced in Section V.

IV. Bounding the Parameters of the Linear Dynamic Model

In the second stage of our procedure, we evaluate bounds on the parameters of the linear dynamic block. Given the exact description of the feasible parameter set $\mathcal{D}_\gamma$, tight bounds on the inner unmeasurable signal x_t can be computed for all inputs u_t belonging to a transient sequence $\{u_t\}$, through the following expressions

$$x_t^{\min} = \min_{\gamma\in\mathcal{D}_\gamma} \varphi_t^{\mathrm{T}}\gamma \quad x_t^{\max} = \max_{\gamma\in\mathcal{D}_\gamma} \varphi_t^{\mathrm{T}}\gamma, \qquad t = 1, 2, \dots, N \tag{17}$$

where $\varphi_t = [\psi_1(u_t)\ \psi_2(u_t)\ \psi_3(u_t)\dots\psi_n(u_t)]^{\mathrm{T}}$. Computation of bounds in (17) requires the solution of $2N$ LP problems with n variables and $2M$ constraints.

Remark: A similar approach is taken by Belforte and Gay [25] who propose the computation of bounds on x_t in order to refine the evaluation of parameter uncertainty intervals of the linear system. However, the bounds they compute are not guaranteed to be tight since their evaluation is based on outer approximations of the nonlinear block parameter set. On the contrary, (17) of this work provides tight bounds on $x_t^{\min}$ and $x_t^{\max}$ since they are computed on the basis of $\mathcal{D}_\gamma$ which is an exact description of the nonlinear block parameter region.

If we define the following quantities:

$$x_t^c = \frac{x_t^{\min} + x_t^{\max}}{2} \quad \Delta x_t = \frac{x_t^{\max} - x_t^{\min}}{2} \tag{18}$$

a compact description of x_t in terms of its central value x_t^c and its perturbation δx_t is as follows:

$$x_t = x_t^c + \delta x_t \tag{19}$$

$$|\delta x_t| \le \Delta x_t. \tag{20}$$

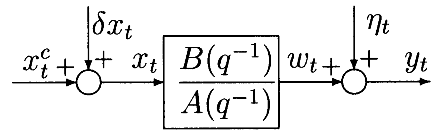

Fig. 3. Errors-in-variables setup for bounding the parameters of the linear system.

We can now formulate the identification of the linear model in terms of the noisy output sequence $\{y_t\}$ and the uncertain inner sequence $\{x_t\}$ as shown in Fig. 3. Such a formulation is commonly referred to as an errors-in-variables (EIV) problem, i.e., a parameter estimation problem in a linear-in-parameter model where the output and some or all the explanatory variables are uncertain.

As a matter of fact, combining (2)–(5) and (19), we get

$$y_t = -\sum_{i=1}^{na}(y_{t-i} - \eta_{t-i})a_i + \sum_{j=0}^{nb}\left(x_{t-j}^c + \delta x_{t-j}\right)b_j + \eta_t. \tag{21}$$

The definition of the feasible parameter region for the linear system is

$$\begin{aligned}\mathcal{D}_\theta = \big\{\theta \in R^p : A(q^{-1})[y_t - \eta_t] = B(q^{-1})\,[x_t^c + \delta x_t]\,;\\ g = 1; |\eta_t| \le \Delta\eta_t; |\delta x_t| \le \Delta x_t,\, t = 1, \dots, N\big\}\end{aligned} \tag{22}$$

where $g = 1$ takes account of condition (9) on the steady-state gain. From (21) it can be seen that consecutive regressions are related deterministically by uncertain output samples and uncertain input samples; that occurrence qualifies the problem as a dynamic EIV. It is referred to as a static EIV problem when the uncertain variables appearing in successive regressions are supposed to vary independently. The relations between successive regressions in the dynamic EIV case give rise to possibly nonlinear exact parameter bounds, which could be not easily and exactly computed [32]. On the other end, in the static EIV case exact parameter bounds are piecewise linear and, although generally non convex, the feasible parameter region is the union of at most 2^p convex sets: each being the intersection of the FPS with a single orthant of the p-dimensional parameter space (a detailed discussion on the geometrical and topological structure of the feasible parameter region for static EIV problems can be found in [33]). Thus, as shown in [32], the FPS of static EIV can be more conveniently handled than the FPS of dynamic EIV. That motivates the use, in this note, of results from the static EIV [33]; since in model (21) the uncertain variables appearing in successive regressions are deterministically related, only outer approximations of the exact feasible parameter region will be obtained. Thus, in this work, a polytopic outer approximation $\mathcal{D}'_\theta$ of the exact FPS $\mathcal{D}_\theta$, i.e. $\mathcal{D}'_\theta \supset \mathcal{D}_\theta$, will be presented, together with an orthotope-outer bounding set $\mathcal{B}_\theta$ of $\mathcal{D}'_\theta$, which provides parameter uncertainties intervals. When we apply results from [33] to our problem we get the following description of the feasible parameter set $\mathcal{D}'_\theta$ at the single time t

$$(\phi_t - \Delta\phi_t)^{\mathrm{T}}\theta \le y_t + \Delta\eta_t \quad (\phi_t + \Delta\phi_t)^{\mathrm{T}}\theta \ge y_t - \Delta\eta_t \tag{23}$$

$$[1 \quad \dots \quad 1 \quad -1 \quad -1 \quad \dots \quad -1]\,\theta = -1 \tag{24}$$

where

$$\phi_t^{\mathrm{T}} = [-y_{t-1}\dots - y_{t-na}\, x_t^c\, x_{t-1}^c \dots x_{t-nb}^c] \tag{25}$$

$$\begin{aligned}\Delta\phi_t^{\mathrm{T}} = [&\Delta\eta_{t-1}\mathrm{sgn}(a_1)\ \dots\ \Delta\eta_{t-na}\mathrm{sgn}(a_{na})\ \Delta x_t\mathrm{sgn}(b_0)\\ &\Delta x_{t-1}\mathrm{sgn}(b_1)\ \dots\ \Delta x_{t-nb}\mathrm{sgn}(b_{nb})].\end{aligned} \tag{26}$$

Equation (24) takes account of condition (9) on the steady-state gain. The orthotope-outer bounding set $\mathcal{B}_\theta$ is defined as

$$\mathcal{B}_\theta = \left\{\theta \in R^p : \theta_j = \theta_j^c + \delta\theta_j, |\delta\theta_j| \le \Delta\theta_j, j = 1, \dots, p\right\} \tag{27}$$

where

$$\theta_j^c = \frac{\theta_j^{\min} + \theta_j^{\max}}{2} \quad \Delta\theta_j = \frac{\left|\theta_j^{\max} - \theta_j^{\min}\right|}{2} \tag{28}$$

$$\theta_j^{\min} = \min_{\theta \in \mathcal{D}'_\theta} \theta_j \quad \theta_j^{\max} = \max_{\theta \in \mathcal{D}'_\theta} \theta_j. \tag{29}$$

Parameter vectors γ^c and θ^c are Chebishev centers in the ℓ_∞ norm of $\mathcal{D}_\gamma$ and $\mathcal{D}'_\theta$ respectively and are commonly referred to as central estimates. The computational aspects related to the evaluation of the orthotope-outer bounding set $\mathcal{B}_\theta$ are briefly discussed in the following subsection.

Computation of $\mathcal{B}_\theta$: In principle, the computation of $\theta_j^{\min}$ and $\theta_j^{\max}$, $j = 1, \ldots, p$, which define $\mathcal{B}_\theta$, requires the solution of $2p2^p$ LP problems (the coefficient $2p$ accounts for p minimization problems and p maximization problems while 2^p is the number of orthant in the p-dimensional parameter space in which the above $2p$ optimization problems must be carried out) with $2N + p + 1$ constraints ($2N$ constraints derive from (23) with $t = 1, \ldots, N$; p is the number of constraints defining the orthant in the parameter space; last, there is an equality constraint derived from the steady-state gain normalization condition (9)). In practice, however, the computational load can be significantly reduced if the signs of θ_j, $j = 1, \ldots, p$ are a-priori known. Indeed, in that case the number of LP problems dramatically decreases to $2p$. If not available, information about the signs of θ_j can be achieved through a point estimate (using least squares estimates, for example) which will indicate the orthant where the optimization should be carried out. If the obtained $\mathcal{B}_\theta$ is such that some of $\theta_j^{\min}$ ($\theta_j^{\max}$) are zero, then the optimization problems should be also solved in the orthants characterized by $\theta_j < 0$ ($\theta_j > 0$).

As to the computational complexity of methods for solving LP problems, it is well known that the ellipsoidal algorithm proposed by Khachiyan [34] seems not to perform satisfactorily, although it shows a worst case polynomial complexity. On the contrary, the simplex method which is widely used in practice and for which Klee and Minty [35] constructed pathological examples that clearly prove its worst case exponential complexity, performs quite well *on the average* (see, e.g., [36]). Consequently, the lack of a polynomial bound on the simplex method is more of theoretical interest than of practical one.

V. Simulated Example

In this section, we illustrate the proposed parameter bounding procedure through a numerical example. The system considered here is characterized by (1), (2), and (5) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$, $\psi_1(u_t) = u_t$; $\psi_2(u_t) = u_t^2$; $\psi_3(u_t) = u_t^3$; $A(q^{-1}) = (1 - 1.1q^{-1} + 0.28q^{-2})$ and $B(q^{-1}) = (0.1q^{-1} + 0.08q^{-2})$. Thus, the true parameter vectors are $\gamma = [\gamma_1\ \gamma_2\ \gamma_3]^{\mathrm{T}} = [1\ 1\ 1]^{\mathrm{T}}$ and $\theta = [a_1\ a_2\ b_1\ b_2]^{\mathrm{T}} = [-1.1\ 0.28\ 0.1\ 0.08]^{\mathrm{T}}$. Two different structures of measurement errors are considered: relative and absolute error. From the simulated transient sequence $\{w_t, \eta_t\}$ and steady-state data $\{\bar{w}_s, \bar{\eta}_s\}$, the SNR and $\overline{\mathrm{SNR}}$ are evaluated, respectively, through

$$\mathrm{SNR} = 10 \log \left\{ \frac{\sum_{t=1}^{N} w_t^2}{\sum_{t=1}^{N} \eta_t^2} \right\}, \overline{\mathrm{SNR}} = 10 \log \left\{ \frac{\sum_{s=1}^{M} \bar{w}_s^2}{\sum_{s=1}^{M} \bar{\eta}_s^2} \right\}. \tag{30}$$

A. Relative Errors

First, bounded relative output errors have been considered when simulating the collection of both steady state and transient data. More precisely, we assumed $\eta_t = \epsilon_t^y y_t$, $|\epsilon_t^y| \leq \Delta\epsilon_t^y$, $\bar{\eta}_s = \bar{\epsilon}_s^y \bar{y}_s$, $|\bar{\epsilon}_s^y| \leq \Delta\bar{\epsilon}_s^y$; where $\{\epsilon_t^y\}$ and $\{\bar{\epsilon}_s^y\}$ are random sequences belonging to the uniform distributions $U[-\Delta\epsilon_t^y, +\Delta\epsilon_t^y]$ and $U[-\Delta\bar{\epsilon}_s^y, +\Delta\bar{\epsilon}_s^y]$ respectively. Bounds on steady-state and transient output measurement errors were supposed to have the same value, i.e., $\Delta\epsilon_t^y = \Delta\bar{\epsilon}_s^y \triangleq \Delta\epsilon^y$. Five different values of uncertainty bounds were considered: $\Delta\epsilon^y = 0.1\%$, 1%, 5%, 10%, 20%. For a given $\Delta\epsilon^y$, the length of steady-state and the transient data are $M = 10$ and $N = [100, 1000]$ respectively. The steady-state input sequence $\{\bar{u}_s\}$ belongs to the interval $[-2, +2]$, while the transient input sequence $\{u_t\}$ belongs to the uniform distribution $U[-2, +2]$. Results about the nonlinear and the linear block are reported in Tables I, III, and V, respectively. For low noise level

TABLE I
Relative Error—Nonlinear Block Parameter Central Estimates (γ_j^c) and Parameter Uncertainty Bounds ($\Delta\gamma_j$) Against Varying Measurements Uncertainty ($\Delta\epsilon^y$) and Signal-to-Noise Ratio ($\overline{\mathrm{SNR}}$)

$\Delta\epsilon^y$ (%)	$\overline{SNR}$ (dB)	γ_j	True Value	γ_j^c	$\Delta\gamma_j$
0.1	61.4	γ_1	1.000	1.000	7.1e-4
		γ_2	1.000	1.000	7.8e-4
		γ_3	1.000	0.999	8.7e-4
1	43.8	γ_1	1.000	0.997	7.4e-3
		γ_2	1.000	1.001	6.6e-3
		γ_3	1.000	1.000	6.3e-3
5	38.1	γ_1	1.000	1.019	3.4e-2
		γ_2	1.000	0.983	6.0e-2
		γ_3	1.000	0.987	4.9e-2
10	25.7	γ_1	1.000	1.059	1.1e-1
		γ_2	1.000	0.950	1.3e-1
		γ_3	1.000	0.919	1.2e-1
20	17.6	γ_1	1.000	0.997	1.0e-1
		γ_2	1.000	0.994	1.1e-1
		γ_3	1.000	0.961	1.3e-1

TABLE II
Absolute Error—Nonlinear Block Parameter Central Estimates (γ_j^c) and Parameter Uncertainty Bounds ($\Delta\gamma_j$) Against ($\overline{\mathrm{SNR}}$)

$\overline{SNR}$ (dB)	γ_j	True Value	γ_j^c	$\Delta\gamma_j$
58.9	γ_1	1.000	1.001	5.1e-3
	γ_2	1.000	0.998	2.1e-3
	γ_3	1.000	1.000	2.3e-3
49.3	γ_1	1.000	1.016	2.7e-2
	γ_2	1.000	0.996	5.7e-3
	γ_3	1.000	0.996	9.8e-3
42.4	γ_1	1.000	0.964	1.0e-1
	γ_2	1.000	1.008	2.1e-2
	γ_3	1.000	1.007	3.5e-2
31.7	γ_1	1.000	1.085	1.7e-1
	γ_2	1.000	1.003	7.1e-2
	γ_3	1.000	0.979	7.3e-2
19.7	γ_1	1.000	1.344	5.2e-1
	γ_2	1.000	0.899	1.7e-1
	γ_3	1.000	0.945	2.1e-1

TABLE III
RELATIVE ERROR—LINEAR SYSTEM PARAMETER CENTRAL ESTIMATES (θ_j^c) AND PARAMETER UNCERTAINTY BOUNDS ($\Delta\theta_j$) AGAINST VARYING MEASUREMENTS UNCERTAINTY ($\Delta\epsilon^y$) AND SNR WHEN $N = 100$

$\Delta\epsilon^y$ (%)	SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
0.1	64.8	θ_1	-1.100	-1.100	1.7e-3
		θ_2	0.280	0.280	1.6e-3
		θ_3	0.100	0.100	1.3e-4
		θ_4	0.080	0.080	1.6e-4
1	44.6	θ_1	-1.100	-1.103	1.7e-2
		θ_2	0.280	0.282	1.5e-2
		θ_3	0.100	0.100	1.3e-3
		θ_4	0.080	0.079	3.0e-3
5	31.3	θ_1	-1.100	-1.095	6.6e-2
		θ_2	0.280	0.275	5.6e-2
		θ_3	0.100	0.099	8.4e-3
		θ_4	0.080	0.080	1.3e-2
10	24.7	θ_1	-1.100	-1.135	1.2e-1
		θ_2	0.280	0.315	1.1e-1
		θ_3	0.100	0.106	1.6e-2
		θ_4	0.080	0.083	2.1e-2
20	18.8	θ_1	-1.100	-1.181	3.7e-1
		θ_2	0.280	0.370	3.2e-1
		θ_3	0.100	0.101	2.6e-2
		θ_4	0.080	0.079	5.2e-2

TABLE IV
ABSOLUTE ERROR—LINEAR SYSTEM PARAMETER CENTRAL ESTIMATES (θ_j^c) AND PARAMETER UNCERTAINTY BOUNDS ($\Delta\theta_j$) AGAINST SNR WHEN $N = 100$

SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
60.0	θ_1	-1.100	-1.100	4.2e-3
	θ_2	0.280	0.280	3.6e-3
	θ_3	0.100	0.100	4.2e-4
	θ_4	0.080	0.080	5.8e-4
47.8	θ_1	-1.100	-1.099	1.5e-2
	θ_2	0.280	0.278	1.4e-2
	θ_3	0.100	0.101	1.7e-3
	θ_4	0.080	0.080	2.2e-3
36.1	θ_1	-1.100	-1.095	5.5e-2
	θ_2	0.280	0.278	5.1e-2
	θ_3	0.100	0.100	4.8e-3
	θ_4	0.080	0.081	5.5e-3
24.8	θ_1	-1.100	-1.096	1.1e-1
	θ_2	0.280	0.281	1.1e-1
	θ_3	0.100	0.103	8.7e-3
	θ_4	0.080	0.084	1.8e-2
13.8	θ_1	-1.100	-1.366	4.5e-1
	θ_2	0.280	0.526	4.5e-1
	θ_3	0.100	0.092	5.5e-2
	θ_4	0.080	0.067	6.6e-2

TABLE V
RELATIVE ERROR—LINEAR SYSTEM PARAMETER CENTRAL ESTIMATES (θ_j^c) AND PARAMETER UNCERTAINTY BOUNDS ($\Delta\theta_j$) AGAINST VARYING MEASUREMENTS UNCERTAINTY ($\Delta\epsilon^y$) AND SNR WHEN $N = 1000$

$\Delta\epsilon^y$ (%)	SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
0.1	64.9	θ_1	-1.100	-1.100	3.3e-4
		θ_2	0.280	0.280	3.3e-4
		θ_3	0.100	0.100	6.8e-5
		θ_4	0.080	0.080	6.7e-5
1	44.9	θ_1	-1.100	-1.099	3.5e-3
		θ_2	0.280	0.279	3.2e-3
		θ_3	0.100	0.100	4.9e-4
		θ_4	0.080	0.080	8.7e-4
5	31.0	θ_1	-1.100	-1.103	1.9e-2
		θ_2	0.280	0.282	1.8e-2
		θ_3	0.100	0.101	5.0e-3
		θ_4	0.080	0.081	5.1e-3
10	24.8	θ_1	-1.100	-1.123	4.4e-2
		θ_2	0.280	0.302	4.8e-2
		θ_3	0.100	0.107	1.0e-2
		θ_4	0.080	0.081	1.1e-2
20	18.8	θ_1	-1.100	-1.103	7.6e-2
		θ_2	0.280	0.284	7.6e-2
		θ_3	0.100	0.105	1.2e-2
		θ_4	0.080	0.081	1.4e-2

TABLE VI
ABSOLUTE ERROR—LINEAR SYSTEM PARAMETER CENTRAL ESTIMATES (θ_j^c) AND PARAMETER UNCERTAINTY BOUNDS ($\Delta\theta_j$) AGAINST SNR WHEN $N = 1000$

SNR (dB)	θ_j	True Value	θ_j^c	$\Delta\theta_j$
59.9	θ_1	-1.100	-1.100	2.1e-3
	θ_2	0.280	0.280	2.0e-3
	θ_3	0.100	0.100	2.5e-4
	θ_4	0.080	0.080	3.4e-4
50.0	θ_1	-1.100	-1.101	7.0e-3
	θ_2	0.280	0.281	6.5e-3
	θ_3	0.100	0.100	7.7e-4
	θ_4	0.080	0.080	1.2e-3
39.9	θ_1	-1.100	-1.094	2.5e-2
	θ_2	0.280	0.276	2.6e-2
	θ_3	0.100	0.100	3.2e-3
	θ_4	0.080	0.080	4.7e-3
29.7	θ_1	-1.100	-1.096	7.2e-2
	θ_2	0.280	0.274	6.4e-2
	θ_3	0.100	0.099	4.8e-3
	θ_4	0.080	0.081	1.3e-2
20.2	θ_1	-1.100	-1.195	3.2e-1
	θ_2	0.280	0.358	2.8e-1
	θ_3	0.100	0.105	2.1e-2
	θ_4	0.080	0.069	4.9e-2

($\Delta\epsilon^y = 0.1\%$) and for all N, the central estimates of both the nonlinear static block and the linear model are consistent with the true parameters. For higher noise level ($\Delta\epsilon^y \geq 1\%$), both γ^c and θ^c give satisfactory estimates of the true parameters. As the number of observations increases (from $N = 100$ to $N = 1000$), parameter uncertainty bounds $\Delta\gamma_j$ and $\Delta\theta_j$ decreases unsurprisingly.

B. Absolute Errors

Next, bounded absolute output errors have been considered when simulating the collection of both steady state data, $\{\bar{u}_s, \bar{y}_s\}$, and transient sequence $\{u_t, y_t\}$. Here, we assumed $|\eta_t| \leq \Delta\eta_t$ and $|\bar{\eta}_s| \leq \Delta\bar{\eta}_s$ where η_t and $\bar{\eta}_s$, are random sequences belonging to the uniform distributions $U[-\Delta\eta_t, +\Delta\eta_t]$ and $U[-\Delta\bar{\eta}_s, +\Delta\bar{\eta}_s]$ respectively. Bounds on steady-state and transient output measurement errors were supposed to have the same value, i.e., $\Delta\eta_t = \Delta\bar{\eta}_s \triangleq \Delta\eta$, and were chosen in such a way as to simulate five different values of signal to noise ratio at the output, namely 60, 50, 40, 30, and 20 dB. For a given $\Delta\eta$, the length of steady-state and the transient data are $M = 10$ and $N = [100, 1000]$, respectively. The steady-state input sequence $\{\bar{u}_s\}$ belongs to the interval $[-2, +2]$, while the transient input sequence $\{u_t\}$ belongs to the uniform distribution $U[-2, +2]$. Results about the nonlinear and the linear block are reported in Tables II, IV, and VI, respectively. For low noise level (SNR = 60 dB) and for all N, the central estimates of both the nonlinear static block and the linear model are consistent with the true parameters. For higher noise level $(\mathrm{SNR} \leq 40\ \mathrm{dB})$, both γ^c and θ^c give satisfactory estimates of the true parameters. As the number of observations increases (from $N = 100$ to $N = 1000$), parameter uncertainty bounds $\Delta\gamma_j$ and $\Delta\theta_j$ decreases, as expected.

VI. Conclusion

A two-stage parameter bounding procedure for SISO Hammerstein models for systems with bounded output errors has been outlined. First, using steady-state input-output data, parameters of the nonlinear block, which was assumed to be modeled by a linear combination of a finite and known number of nonlinear static functions, have been tightly bounded. Then, for a given input transient sequence we have computed bounds on the unmeasurable inner signal which, together with output noisy measurements have been used to overbound the parameters of the linear part. The numerical example showed the effectiveness of the proposed procedure.

Acknowledgment

The authors would like to thank the reviewers for their constructive comments and suggestions.

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